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THE BIFURCATION AND STABILITY OF PERMANENT ROTATIONS OF A HEAVY TRIAXIAL ELLIPSOID ON A SMOOTH PLANE*

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Stationary motions of a heavy triaxial ellipsoid on a smooth horizontal plane are investigated. Permanent rotations of the ellipsoid are determined as well as the conditions for their existence, branching and stability. Certain special features of the problem in question are noted, namely the loss of secular stability of the rapid rotation of the ellipsoid when its centre of mass is at its lowest, the increase in dimensionality of the manifold of permanent rotations of the ellipsoid in the problem when the necessary conditions for the existence of the additional integral hold in this case, etc. Earlier results obtained by the authors concerning the stability of regular precessions of a top on a horizontal plane with friction, are made more precise.

1. Consider a heavy rigid body on a small horizontal plane assuming, without loss of generality, that the projection of the centre of mass of the body on the supporting plane, is stationary. We shall assume that the body is bounded by an ellipsoidal surface whose principal axes coincide with the principal axes of the central ellipsoid of inertia.

The equations of motion of the body admit of the first integrals

$$U = m\dot{z}^2 + J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 + 2mgz = \text{const} \tag{1.1}$$

$$U_1 = J_1\omega_1\gamma_1 + J_2\omega_2\gamma_2 + J_3\omega_3\gamma_3 = k = \text{const} \tag{1.2}$$

$$U_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \quad z^2 = a_1^2\gamma_1^2 + a_2^2\gamma_2^2 + a_3^2\gamma_3^2 \tag{1.3}$$

Here z is the height of the centre of mass above the supporting plane, $\omega_1, \omega_2, \omega_3$ and $\gamma_1, \gamma_2, \gamma_3$ are the components of the angular velocity vector of the body and the unit vector of the ascending vertical in the principal central axes of inertia of the body, m is its mass, J_1, J_2, J_3 and a_1, a_2, a_3 are the principal central moments of inertia of the ellipsoid and the semi-axes of its surface, and g is the acceleration due to gravity. Confining ourselves to investigating the triaxial ellipsoids with triaxial ellipsoids of inertia we will assume, without loss of generality, that $J_1 < J_2 < J_3$.

According to Routh's theorem the stationary motions of the system in question have the corresponding stationary values of the energy integral (1.1) when the values of the area (1.2) and geometrical (1.3) integrals are constant. We recuce the problem of determining these motions to the problem of determining the stationary values of the function

$$2W = U - 2\lambda(U_1 - k) + \sigma(U_2 - 1)$$

where λ and σ are the undetermined Lagrange multipliers.

The function W depending on the variables $\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3, \lambda, \sigma$ and parameter k, is an analogue of the modified potential energy /1/.

The conditions of stationarity of the function W lead to the equations

$$\frac{\partial W}{\partial \lambda} = k - U_1 = 0, \quad \frac{\partial W}{\partial \sigma} = \frac{1}{2}(U_2 - 1) = 0, \quad \frac{\partial W}{\partial \dot{z}} = m\dot{z} = 0 \tag{1.4}$$

$$\frac{\partial W}{\partial \omega_1} = J_1(\omega_1 - \lambda\gamma_1) = 0, \quad \frac{\partial W}{\partial \gamma_1} = \left(\frac{mg}{z}a_1^2 + \sigma\right)\gamma_1 - \lambda J_1\omega_1 = 0 \quad (123) \tag{1.5}$$

Here and henceforth the symbol (123) will indicate that the relations omitted are obtained by circular permutation of the indices 1, 2, 3.

Depending on the relations between the semi-axes of the surface of the body, the last equations admit of between three and six one-parameter families of solutions (apart from the signs of $\gamma_1, \gamma_2, \gamma_3$) of the form

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$$z^{\cdot}=0,\quad \omega_1=\lambda\gamma_1,\quad \gamma_1=\gamma_1(\lambda)\quad (123);\quad \sigma=\sigma(\lambda),\quad k=k(\lambda),\quad \lambda=\mathrm{const}$$

corresponding to the uniform rotations of the body about the vertical passing through the fixed centre of mass of the body.

2. Let $a_1<a_2<a_3$. Then Eqs. (1.4) and (1.5) will have six one-parameter families of solutions (the relation $z^{\cdot}=0$, common to all these solutions is omitted)

$$\omega_1=\lambda\gamma_1,\quad \omega_2=\omega_3=0,\quad \gamma_1^2=1,\quad \gamma_2=\gamma_3=0 \tag{2.1}$$
$$\sigma=J_1\lambda^2-mga_1,\quad k=J_1\lambda\quad (123)$$

$$\omega_1=0,\quad \omega_2=\lambda\gamma_2,\quad \omega_3=\lambda\gamma_3,\quad \gamma_1=0 \tag{2.2}$$
$$\gamma_2^2=\frac{a_3^2(\lambda^4-\lambda_{12}^4)}{\lambda^4(a_3^2-a_2^2)},\quad \gamma_3^2=\frac{a_2^2(\lambda_{13}^4-\lambda^4)}{\lambda^4(a_3^2-a_2^2)}$$
$$\sigma=\frac{J_2a_3^2-J_3a_2^2}{a_3^2-a_2^2}\lambda^2,\quad k=k_1(\lambda)=J_{23}\lambda\quad (123)$$
$$\lambda_{12}^2=\frac{mg(a_3^2-a_2^2)}{a_3(J_3-J_2)},\quad \lambda_{13}^2=\frac{mg(a_3^2-a_2^2)}{a_2(J_3-J_2)}\quad (123)$$
$$J_{23}=\frac{J_2a_3^2-J_3a_2^2}{a_3^2-a_2^2}+\frac{m^2g^2(a_3^2-a_2^2)}{\lambda^4(J_3-J_2)}\quad (123)$$

The solutions (2.1) and (2.2) describe the uniform rotations of the body with angular velocity λ about the corresponding principal central axes of inertia and the axes lying in the principal planes of the central ellipsoid of inertia. We note that solutions (2.1) exist for any value of the angular velocity λ, and solutions (2.2) for

$$\lambda_{12}^2\leqslant\lambda^2\leqslant\lambda_{13}^2\ (1);\quad \lambda_{23}^2\geqslant\lambda^2\geqslant\lambda_{21}^2\ (2);\quad \lambda_{31}^2\leqslant\lambda^2\leqslant\lambda_{32}^2\ (3) \tag{2.3}$$

respectively.

The solutions (2.1) and (2.2) can be represented geometrically in a nine-dimensional space $(\omega_1,\omega_2,\omega_3,\gamma_1,\gamma_2,\gamma_3,\lambda,\sigma,k)$ by the points of a one-dimensional curve whose six branches P_1,P_2,P_3 and Q_1,Q_2,Q_3 are determined, respectively, by Eqs. (2.1) and (2.2).

The branches P_1,P_2 and P_3 determined by relations (2.1) correspond to the uniform rotations of the body about the principal axes of inertia and represent, in the subspace (λ,σ,k), the parabolas p_1,p_2,p_3 with a common axis of symmetry situated, **respectively**, in the plane $\pi_1\ (k=J_1\lambda)$, $\pi_2\ (k=J_2\lambda)$, $\pi_3\ (k=J_3\lambda)$. The positions of equilibrium of the body correspond to the spaces of the parabolas.

The branches Q_1,Q_2 and Q_3, determined from relations (2.2) correspond to the uniform rotations of the body about axes lying in the principal planes of inertia and represent, in the subspace (λ,σ,k), the segments of the curves q_1,q_2,q_3, situated between the planes π_2 and π_3, π_3, and π_1, π_1 and π_2 respectively; the ends of these segments lie, respectively, on the parabolas p_2 and p_3, p_3, and p_1, p_1 and p_2.

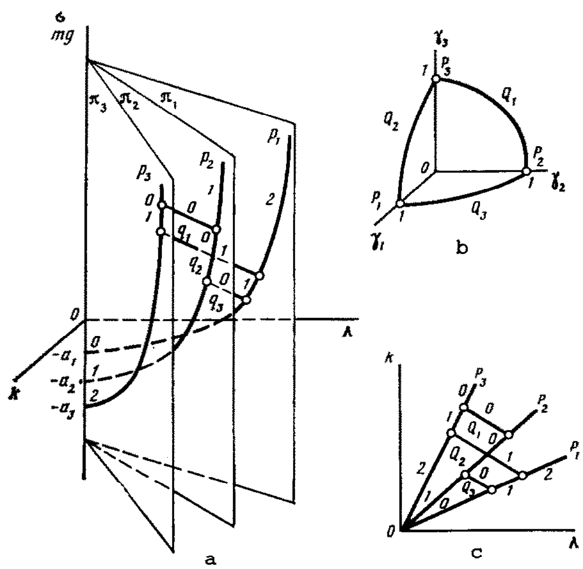

Fig.1

The branches P_1, P_2, P_3 correspond, in the subspace $(\gamma_1, \gamma_2, \gamma_3)$, to the points $\gamma_1^2 = 1$, $\gamma_2^2 = 1$, $\gamma_3^2 = 1$ of intersection of the unit sphere (1.3) with the coordinate axes, and the branches Q_1, Q_2, Q_3 correspond to the large circles passing through the points $\gamma_2^2 = 1$ and $\gamma_3^2 = 1$, $\gamma_3^2 = 1$, and $\gamma_1^2 = 1$, $\gamma_1^2 = 1$ and $\gamma_2^2 = 1$ respectively.

The approximate form of the branches P_1, P_2, P_3 and Q_1, Q_2, Q_3 is shown in Fig.1 (a in the subspace (λ, σ, k) for $\lambda > 0$, $k > 0$, b in the subspace $(\gamma_1, \gamma_2, \gamma_3)$ for $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ c in the plane (λ, k) for $\lambda > 0$ $k > 0$).

3. We will use Routh's theorem and its inversion to investigate the stability of permanent rotations of the body relative to $\omega_1, \omega_2, \omega_3$ and $\gamma_1, \gamma_2, \gamma_3$. We will denote the variations of the variables $\omega_1, \omega_2, \omega_3$ and $\gamma_1, \gamma_2, \gamma_3$ by ξ_1, ξ_2, ξ_3 and η_1, η_2, η_3 respectively. Then the second variation $\delta^2 W$ of the function W will take the following form for the solutions (2.1), under the conditions that $\delta U_1 = \delta U_2 = 0$:

$$\delta^2 W = mz^{\cdot 2} + J_2(\xi_2 - \lambda\eta_2)^2 + J_3(\xi_3 - \lambda\eta_3)^2 + [(J_1 - J_2)\lambda^2 + mga_1^{-1}(a_2^2 - a_1^2)]\eta_2^2 + [(J_1 - J_3)\lambda^2 + mga_1^{-1}(a_3^2 - a_1^2)]\eta_3^2 \quad (123) \tag{3.1}$$

From this it follows that the rotations of the body about the x_1 axis, corresponding to the smallest moment of inertia J_1 are stable (the degree of instability $\chi = 0$), if $0 \leqslant \lambda^2 < \lambda_{32}^2$, unstable $(\chi = 1)$, if $\lambda_{32}^2 < \lambda^2 < \lambda_{23}^2$, and the degree of instability for them will be $\chi = 2$, when $\lambda^2 > \lambda_{23}^2$.

The rotations of the body about the x_2 axis corresponding to the middle moment of inertia J_2 are unstable $(\chi = 1)$ if $0 \leqslant \lambda^2 < \lambda_{31}^2$ or $\lambda^2 > \lambda_{13}^2$, and stable $(\chi = 0)$ when $\lambda_{31}^2 < \lambda^2 < \lambda_{13}^2$.

The rotations of the body about the x_3 axis corresponding to the largest moment of inertia J_3, have the degree of instability $\chi = 2$ when $0 \leqslant \lambda^2 < \lambda_{21}^2$, are unstable $(\chi = 1)$ when $\lambda_{21}^2 < \lambda^2 < \lambda_{12}^2$, and stable $(\chi = 0)$, when $\chi^2 > \lambda_{12}^2$.

The values $\lambda^2 = \lambda_{ij}^2$ $(i, j = 1, 2, 3;\ i \neq j)$ correspond to the bifurcation points. The branches P_1, P_2, P_3 produce, at these points, the branches Q_1, Q_2, Q_3, corresponding to the permenant rotations (2.2).

In the case of the solutions (2.2) the second variation of the function W, under the conditions $\delta U_1 = \delta U_2 = 0$, takes the form

$$\delta^2 W = mz^{\cdot 2} + J_1(\xi_1 - \lambda\eta_1)^2 + \frac{J_2 J_{23}}{J_3 \gamma_3^2}\left[\xi_2 - \lambda\eta_2 - \frac{2\lambda(J_3 - J_2)\gamma_2^2}{J_{23}}\eta_2\right]^2 + \varkappa\frac{\lambda^2}{a_3^2 - a_2^2}\eta_1^2 - \frac{(J_3 - J_2)^3\lambda^6\gamma_2^2}{m^2 g^2 J_{23}(a_3^2 - a_2^2)}\frac{dk_1}{d\lambda}\eta_2^2 \quad (123) \tag{3.2}$$

$$\varkappa = (J_3 - J_2)a_1^2 + (J_1 - J_3)a_2^2 + (J_2 - J_1)a_3^2$$

The analysis of the functions $k_i(\lambda)$ (from now on we have $i = 1, 2, 3$ everywhere) yields:

1°. $dk_i/d\lambda < 0$ for all λ from the corresponding interval of (2.3), provided that

$$\delta_i < 4\varepsilon_i(1 + 3\varepsilon_i)^{-1} \tag{3.3}$$

and also for $\lambda^2 < \lambda_i^2$, if

$$4\varepsilon_i(1 + 3\varepsilon_i)^{-1} \leqslant \delta_i \leqslant (3 + \varepsilon_i)/4 \tag{3.4}$$

2°. $dk_i/d\lambda > 0$ for all λ, if

$$\delta_i > (3 + \varepsilon_i)/4 \tag{3.5}$$

and also for $\lambda^2 > \lambda_i^2$, if δ_i and ε_i satisfy the relation (3.4).

Here we have

$$\lambda_1^4 = \frac{3m^2 g^2 (a_3^2 - a_2^2)^2}{(J_3 - J_2)(J_2 a_3^2 - J_3 a_2^2)} \quad (123)$$

$$\delta_1 = \frac{J_2}{J_3}, \quad \delta_2^{-1} = \frac{J_3}{J_1}, \quad \delta_3 = \frac{J_1}{J_2}; \quad \varepsilon_1 = \frac{a_2^2}{a_3^2}, \quad \varepsilon_2^{-1} = \frac{a_3^2}{a_1^2}, \quad \varepsilon_3 = \frac{a_1^2}{a_2^2}$$

Let $\varkappa > 0$. Then the rotations of the body about axes perpendicular to the x_1 axis are stable $(\chi = 0)$ under the condition (3.3), $i = 1$, and also under the condition (3.4) $i = 1$, if $\lambda^2 < \lambda_1^2$, and unstable $(\chi = 1)$ under the condition (3.5) $i = 1$, and also under the condition (3.4) $i = 1$, if $\lambda^2 > \lambda_1^2$.

The rotations of thebody about axes perpendicular to the x_2 axis are unstable $(\chi = 1)$ the condition (3.3) $i = 2$, and also under the condition (3.4) $i = 2$ if $\lambda^2 < \lambda_2^2$ and their degree of instability is $\chi = 2$ under the condition (3.5) $i = 2$ and also under the condition (3.4) $i = 2$, if $\lambda^2 > \lambda_2^2$.

The rotations of the body about axes perpendicular to the x_3 axis are stable $(\chi = 0)$ under the condition (3.3) $i = 3$, and also under the condition (3.4) $i = 3$, if $\lambda^2 < \lambda_3^2$, and unstable $(\chi = 1)$ under the condition (3.5) $i = 3$, and also under the condition (3.4) $i = 3$, if $\lambda^2 > \lambda_3^2$.

When conditions (3.4) hold, the points of bifurcation correspond to the values $\lambda^2 = \lambda_i^2$. At these points the branches Q_i touch the planes $k = k_i$ and are situated on one side of these planes, and

$$k_1 = \frac{J_2 a_3^2 - J_3 a_2^2}{a_3^2 - a_2^2} \lambda_1 + \frac{m^2 g^2 (a_3^2 - a_2^2)}{(J_3 - J_2)\lambda_1^3} \quad (123)$$

If $\varkappa < 0$, then the degree of instability of the rotations of the body about axes perpendicular to the axes of the smallest and largest (mean) moment of inertia increases (decreases) by one; the case of $\varkappa = 0$ needs a special treatment and will be discussed below.

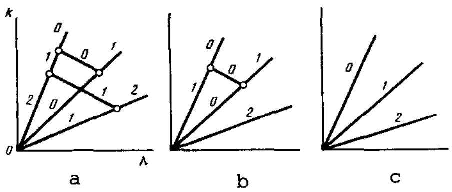

Fig.2

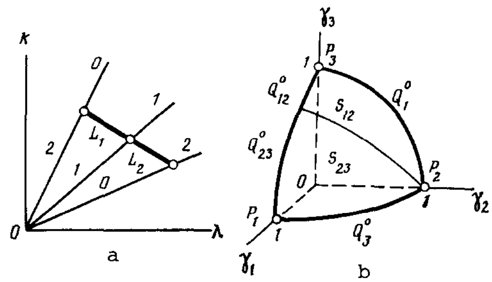

Fig.3

Fig.1a shows a curve of permanent rotations of the body and the distribution of the degree of instability on its branches for $k > 0$ in the case when $\varkappa > 0$ and all conditions of (3.3) hold.

Note. The branches P_i of the rotations of the body about the principal axes of inertia exist for any relations connecting the semi-axes of the surface of the body, while when $a_1 < a_2 < a_3$ does not hold, then some or all branches Q_i of the rotations of the body about the axes lying in its principal planes of inertia vanish (become imaginary). At the same time the number of bifurcation points on the branches P_i decreases and the nature of the stability of the separate segments of these branches, changes.

Thus when $a_2 < a_1 < a_3$ and $\varkappa > 0$), the branch Q_3 vanishes, while when $a_1 < a_3 < a_2$ $(\varkappa < 0)$, then the branch Q_1 vanishes; when $a_2 < a_3 < a_1$ $(\varkappa > 0)$, the branches Q_2 and Q_3 vanish and when $a_3 < a_1 < a_2$ $(\varkappa < 0)$, then the branches Q_1 and Q_2 vanish. Finally, when $a_3 < a_2 < a_1$, which holds a priori for a homogeneous ellipsoid (we recall that $J_1 < J_2 < J_3$ by definition), all branches Q_i vanish.

Fig.2 (we use the notation Fig.1) shows the projections of the curves of permanent rotations of the body on the plane (k, λ) and the distribution of the degree of instability on its branches for $k > 0$ in the cases a) $a_2 < a_1 < a_3$, b) $a_2 < a_3 < a_1$, c) $a_3 < a_2 < a_1$ respectively (when conditions (3.3) are satisfied).

We note that the permanent rotations of the body whose degree of instability is equal to two, can be Lyapunov stable (we also have a gyroscopic stabilization, which however breaks down under the action of forces with dissipation, total with respect to the velocities of position coordinates). A rigorous investigation of the stability of such rotations requires the application of the Kolmogorov-Arnol'd-Moser methods (note that the Routh-reduced system has two degrees of freedom in the present problem), and is not discussed here.

4. Let us now deal with the case $\varkappa = 0$. First we note that this case is possible only when $a_1 < a_2 < a_3$, and the expressions

$$\frac{J_2 a_3^2 - J_3 a_2^2}{a_3^2 - a_2^2}, \quad \frac{J_3 - J_2}{a_3^2 - a_2^2} \quad (123)$$

are invariant for this case under cyclic permutation of the indices 1, 2, 3 (we shall denote them by J and μ respectively; clearly, $\mu > 0$ and $J + \mu z^2 = J_1\gamma_1^2 + J_2\gamma_2^2 + J_3\gamma_3^2 > 0$). Here the Eqs.(1.4), (1.5) admit of all six one-parameter families of solutions (2.1), (2.2) and also a two-parameter family

$$\begin{gathered}\omega_1 = \lambda\gamma_1 \ (123), \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \quad a_1^2\gamma_1^2 + a_2^2\gamma_2^2 + \\ a_3^2\gamma_3^2 = z^2, \quad z = mg/(\mu\lambda^2), \quad \sigma = J\lambda^2, \quad k = J\lambda + m^2g^2/(\mu\lambda^3)\end{gathered} \quad (4.1)$$

(the free parameters are e.g. λ and γ_2).

The solutions (4.1) describe uniform rotations of the body about an arbitrarily positioned vertical axis, and exist when $a_1 \leqslant z \leqslant a_3$, i.e. when

$$mg/(\mu a_3) \leqslant \lambda^2 \leqslant mg/(\mu a_1)$$

The solutions (4.1) can be represented in the space $(\omega_1, \omega_2, \omega_3, \gamma_1, \gamma_2, \gamma_3, \lambda, \sigma, k)$ in the form of a two-dimensional surface S stretched over the branches Q_i° into which the branches Q_i are transformed when $\varkappa = 0$.

In the subspace (λ, σ, k) and surface S and the branches Q_i° merge to form a single multiple curve situated between the planes π_1 and π_3 and intersecting the plane π_2 (Fig.3a shows its projection $(L) = (L_1) \bigcup (L_2)$ on the $(k, \lambda))$ plane, while in the subspace γ_i the branches Q_i° do not coincide and have, as before, the form shown in Fig.1b.

Fig.3b shows the form of the surface S in the positive octant of this subspace. We note that the surface can be split into two parts, S_{12} $(a_2^2 \leqslant z^2 \leqslant a_3^2)$ and S_{23} $(a_1^2 \leqslant z^2 \leqslant a_2^2)$, and their common boundary (the separatrix) will be given by the system of equations

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \quad a_1^2\gamma_1^2 + a_2^2\gamma_2^2 + a_3^2\gamma_3^2 = a_2^2$$

while its point of intersection with the branch Q_2° will also divide the latter into two parts, Q_{12}° $(\lambda^2 \leqslant \lambda_*^2)$ and Q_{23}° $(\lambda^2 \geqslant \lambda_*^2)$, with $\lambda_*^2 = mg/(\mu a_2)$. In the region S_{12} the permanent rotations (4.1) are generated from the permanent rotations about the axes perpendicular to the x_1 axis (Q_1°), and pass continuously to "slow" permanent rotations about axes perpendicular to the x_2 axis (Q_{12}°), and in the region S_{23} the permanent rotations (4.1) are generated by the "rapid" $(\lambda^2 \geqslant \lambda_*^2)$ permanent rotations about the axes perpendicular to the x_2 axis (Q_{23}°), and pass continuously to the permanent rotations about axes perpendicular to the x_3 axis (Q_3°), so that (see Fig.3a) $(L_1) = (Q_1^\circ) = (Q_{12}^\circ) = (S_{12})$, $(L_2) = (Q_3^\circ) = (Q_{23}^0) = (S_{23})$.

The brackets indicate that we consider the projections of the corresponding curves and surfaces on one or other subspace (the brackets are omitted from the figures).

Thus all points of the branches Q_i° are bifurcation points (formally, this also follows from (3.2), since when $\varkappa = 0$, the second variation of the function W becomes degenerate in the case of permanent rotations (2.2)). Consequently when $\varkappa = 0$, neither the permanent rotations (2.2) (since they all correspond to the bifurcation points), nor the permanent rotations (4.1) (since the dimension of the family of these rotations is greater than the number of the arbitrary constant integrals (1.2) and (1.3)), can be stable with respect to all variables under any conditions.

Moreover we note, that when $\varkappa = 0$, the function W can be reduced to the form

$$W^\circ = mz^{\cdot 2} + J_1\Omega_1^2 + J_2\Omega_2^2 + J_3\Omega_3^2 + 2mgz - \mu\lambda^2 z^2$$
$$\Omega_i = \omega_i - \lambda\gamma_i \quad (i = 1, 2, 3)$$

Clearly, the function W° depends on the combination of Ω_i and of z variable ω_i, γ_i, and the number of these conditions is less than the number of the initial variables. Thus in the present case it makes sense to pose a problem of stability of the uniform rotations (2.2) and (4.1) relative to the variables Ω_i, z i.e. with respect to some of the variables.

Let us denote the variation of the variable z by ζ, and retain the previous notation for the variables appearing in the function W°. Then the second variation of the function W° will, under the conditions that $\delta U_1 = \delta U_2 = 0$, take the following form for the solutions (2.2) and (4.1):

$$\delta^2 W^\circ = mz^{\cdot 2} + J_1\Omega_1^2 + J_2\Omega_2^2 + J_3\Omega_3^2 - \frac{\mu\lambda^2}{J + \mu z^2}\frac{dk}{d\lambda}\zeta^2$$

where, as we already said, $\mu > 0$ and $J + \mu z^2 > 0$, and the function $k(\lambda)$ is given by the last relation of (4.1).

The analysis of the function $k(\lambda)$ shows that $dk/d\lambda < 0$ on all branches Q_i° and the whole of the surface S, provided that $J < 3\mu a_1^2$ on the whole branch Q_1°, on the part Q_{12}° of the branch Q_2°, and on the part S_{12} of the surface S, and also under the condition that $\lambda^2 < 3mg/J$, on the branch Q_3°, on the part Q_{23}° of the branch Q_2° and on the part S_{23} of the surface S, provided that $3\mu a_1^2 \leqslant J < 3\mu a_2^2$, and finally, under the condition that $\lambda^2 < 3mg/J$, on the branch Q_1°, on the part Q_{12}° of the branch Q_2° and on the part S_{12} of the surface S, provided that $3\mu a_2^2 \leqslant J < 3\mu a_3^2$ (otherwise $dk/d\lambda > 0$).

Thus the permanent rotations of the body about the axes whose direction cosines belong to the region $S_{12} \bigcup Q_1^\circ \bigcup Q_{12}^\circ$ of the unit sphere (1.3) are always stable if $J < 3\mu a_2^2$, and also if $\lambda^2 < 3mg/J$, when $3\mu a_2^2 \leqslant J < 3\mu a_3^2$. The permanent rotations of the body about the axes whose direction cosines belong to the region $S_{23} \bigcup Q_{23}^\circ \bigcup Q_3^\circ$ are always stable when $J < 3\mu a_1^2$, and also if $\lambda^2 < 3mg/J$ when $3\mu a_1^2 \leqslant J < 3\mu a_2^2$.

Finally we note that when $\varkappa = 0$, then every pair of bifurcation points on the branches P_1, P_2, P_3 merges into one double point. The uniform rotations of the body about the axis of the smallest (largest) moment of inertia are either stable if $\lambda^2 < mg/\mu a_1$ $(\lambda^2 > mg/(\mu a_3))$, or their degree of instability is equal to two if $\lambda^2 > mg/(\mu a_1)$ $(\lambda^2 < mg/(\mu a_3))$, and the uniform rotations of the body about the axis of the middle moment of inertia are nearly always unstable

($\chi = 1$ when $\lambda^2 \neq mg/(\mu a_2)$).

Fig.3a shows the projection of the manifold of permanent rotations of the body on the (k, λ) plane and the distribution of the degrees of instability on its branches for $k > 0$ (under the condition that $J < 3\mu a_1^2$).

We note that the permanent rotations (2.2) and (4.1) for which $dk/d\lambda > 0$, can also be Lyapunov stable (with respect to some of the variables).

6. Let us note some special features of the problem in question. Firstly, the triaxial ellipsoid on a smooth plane may behave in the same way as a "tippy-top" (e.g. when $J_1 < J_2 < J_3$, $a_1 < a_2 < a_3$). If we place such an ellipsoid so that its centre of mass is at its lowest position and spin it rapidly about the vertical, then ellipsoid will roll over into a position in which its centre of mass will be at its highest, and will rotate about the vertical with an angular velocity less than the initial velocity (Fig.1a,c). We note that such behaviour of a triaxial ellipsoid can be explained, unlike the case of the roll-over of a top (a symmetric body), without taking into account the sliding friction.

Secondly, if the parameters of the ellipsoid satisfy the condition $\varkappa = 0$, then the dimension of the manifold of its stationary motions will be greater than the number of known integrals of the problem in question, different from the energy integral and the trivial integral. It is interesting to note that the condition $\varkappa = 0$ is identical with the necessary condition for the existence of the additional integral in this problem /3/.

Finally, when $\varkappa = 0$, the degree of instability of permanent rotations of the body about the principal axes of inertia changes, during the passage through the bifurcation points, by an even number (Fig.3a). Such a change in stability is caused by the fact that the points are multiple. Every one of these points is obtained by merging together a pair of bifurcation points, simple when $\varkappa \neq 0$ (when $\varkappa = 0$, both Poincaré stability coefficients vanish simultaneously at the bifurcation points).

7. A multiplying factor $dk/d\lambda$, where k is given by relation (2.7') of /4/, was omitted in the conditions of stability (instability) (3.5), ((3.6)) of angular precession of a top in /4/. The exact condition of stability (instability) has the form $(J_1 - J_3)\, dk/d\lambda > 0\ (<0)$. When $J_1 > J_3$, the factor $dk/d\lambda$ is greater than zero and the regular precession of a top is always stable. If, on the other hand, $J_1 < J_3$, then $dk/d\lambda$ can take positive values (and the precessions will be unstable), as well as negative values (in which case the precessions will be stable).

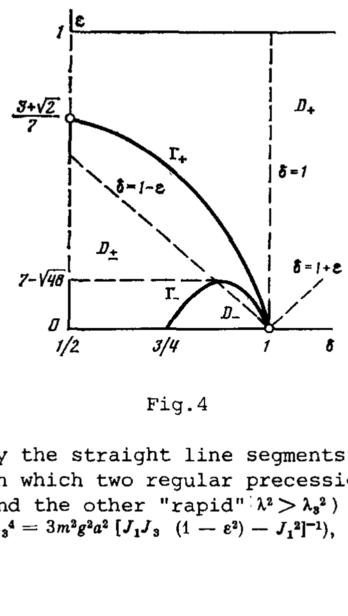

Fig.4

We can separate the following three regions (Fig.4) in the parameter plane $\varepsilon = a/\rho$, $\delta = J_1/J_3$ of the problem (we use the notation of /4/): region D_+ (bounded by the rays $\varepsilon = 0, \delta > 1$ and $\delta = 1/2$, $\varepsilon > (3 + \sqrt{2})/7$ and the curve Γ_+), in which we have at most a single regular precession for any fixed value k in the integral (2.2) /4/, and the precession is always stable; region D_- (bounded by the straight line segment $\varepsilon = 0, 3/4 < \delta < 1$ and the curve Γ_-), in which we also have at most a single regular precession for any value of k and the precession is always unstable; region $D_\pm$ (bounded by the straight line segments $\varepsilon = 0, 1/2 < \delta < 3/4$ and $\delta = 1/2, 0 < \varepsilon < (3 + \sqrt{2})/7$ and curves Γ_+ and Γ_-), in which two regular precessions can exist, one of which will be "slow" $\lambda^2 < \lambda_3^2$) and **stable,** and the other "rapid" $\lambda^2 > \lambda_3^2$) and unstable. Here λ_3 is a real root of the equation $dk/d\lambda = 0$ ($\lambda_3^4 = 3m^2g^2a^2\,[J_1J_3\ (1 - \varepsilon^2) - J_1^2]^{-1}$), and the curves Γ_+ and Γ_- are given, respectively, by the equations

$$\Gamma_+ : 4\delta^2 - \delta\,(1 - \varepsilon)\,(7 + \varepsilon) + 3\,(1 - \varepsilon)^2 = 0$$

$$\Gamma_- : 4\delta^2 - \delta\,(1 + \varepsilon)\,(7 - \varepsilon) + 3\,(1 + \varepsilon)^2 = 0$$

We note that the diagrams a-d shown in the figure in /4/ correspond to values of the parameters lying, respectively, in the region D_- to the left of the straight line $\delta = 1 - \varepsilon$ (a); in the region D_- to the right of the straight line $\delta = 1 - \varepsilon$ (b); in the region D_+ to the right of the straight line $\delta = 1$ and to the left of the straight line $\delta = 1 + \varepsilon$ (c); and in the region D_+ to the right of the straight line $\delta = 1 + \varepsilon$ (d). For the parameters lying in the region D_+ to the left of the stright line $\delta = 1$, the diagram is analogous to that shown in figure b of /4/, except that the whole curvilinear branch must have a plus sign. For the parameters lying in the region $D_\pm$ above (below) the straight line $\delta = 1 - \varepsilon$, the diagram is analogous to that shown in the figure b(a) of /4/, except that the part of the curvilinear branch of this diagram adjacent, when $\lambda^2 < \lambda_3^2$, to the straight line $\gamma_3 = +1$, must also have a plus sign.

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THE EFFECT OF THIRD-AND FOURTH-ORDER MOMENTS OF INERTIA ON THE MOTION OF A SOLID*

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The problem of the effects of higher-order moments of inertia on the motion of a solid, fixed at the centre of mass and having a spherical central ellipsoid of inertia in a central Newtonian field of force is investigated. Uniform bodies of the simplest geometrical shapes (a cube, cone and cylinder) are considered. In view of the difference in the symmetries of these bodies the nature of their motions will be different. The equations of motion of a cone and a cylinder are integrated in terms of ultra-elliptic and hyperelliptic functions respectively. Sets of positions of equilibrium, permanent rotations, and regular precessions are indicated, and their branching and stability are investigated. Unlike the case when only second-order moments of inertia are taken into account, two features are determined here: 1) tow families of inclined positions with respect to equilibrium exist, and 2) for a body in the form of a cone the direct position of relative equilibrium is unstable if the vertex of the cone is situated between an attracting centre and a fixed point, and is stable otherwise, which has no analogue for permanent rotations of a body with a triaxial central ellipsoid of inertia.

1. Suppose $O\xi\eta\zeta$ is a fixed system of coordinates with origin at the centre of mass of a body at a distance R from an attracting centre and an axis ζ directed along a rising local vertical, and $Ox_1x_2x_3$ is a system of coordinates rigidly coupled to the body. The mutual orientation of the ξ, η, ζ and x_1, x_2, x_3 axes is specified by a matrix of direction cosines. We will denote the unit vectors of the ξ, η, ζ axes by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and their projections on to the x_1, x_2, x_3 axes by $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$

The coordinates x_1, x_2, x_3 of a point of the body will be written in dimensionless form by relating them to a characteristic linear dimension a of the body (a is the side of the cube or the radius of the base for a cone and a cylinder).

The force function U of the forces of Newtonian traction has the form (μ is the gravitational constant and ρ is the density of the body)

$$
\begin{gathered}
U = \iiint \frac{\mu\rho}{\Delta}\, dx_1\, dx_2\, dx_3 = \frac{\mu\rho}{R} \iiint f(\varepsilon)\, dx_1\, dx_2,\, dx_3 \\
\Delta = R\,[\varepsilon^2(\xi^2+\eta^2) + (1+\varepsilon\zeta)^2]^{1/2} = R\,[1 + 2\varepsilon\,(x_1\gamma_1 + \\
x_2\gamma_2 + x_3\gamma_3) + \varepsilon^2\,(x_1^2 + x_2^2 + x_3^2)]^{1/2} \quad (\varepsilon = a/R \leqslant 1) \\
f(\varepsilon) = [1 + 2\varepsilon\zeta + \varepsilon^2(\xi^2+\eta^2+\zeta^2)]^{-1/2}
\end{gathered} \tag{1.1}
$$

It can be seen that U is independent of α_i and β_i, and hence equilibrium is preserved as the body rotates about the ζ axis.

We will calculate U up to fourth-order terms in ε using the expansion

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